4.2 Energy and Power

Definition 4.11. For a signal $g(t)$, the instantaneous power $p(t)$ dissipated in the 1-Ω resister is $p_g(t) = |g(t)|^2$ regardless of whether $g(t)$ represents a voltage or a current. To emphasize the fact that this power is based upon unity resistance, it is often referred to as the **normalized (instantaneous) power**.

Definition 4.12. The total (normalized) ***energy*** of a signal $g(t)$ is given by

$$E_g = \int_{-\infty}^{+\infty} p_g(t)dt = \int_{-\infty}^{+\infty} |g(t)|^2\, dt = \lim_{T\to\infty} \int_{-T}^{T} |g(t)|^2\, dt.$$

4.13. By the **Parseval's theorem** discussed in 2.43, we have

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df.$$

ESD : Energy Spectral density

Definition 4.14. The average (normalized) ***power*** of a signal $g(t)$ is given by

"general formula"

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \overbrace{\int_{-T}^{T} |g(t)|^2\, dt}^{\text{energy}}.$$

per unit time

Definition 4.15. To simplify the notation, there are two operators that used angle brackets to define two frequently-used integrals:

(a) The "**time-average**" operator:

$\langle c \rangle = c$

$\langle a(t) + b(t) \rangle = \langle a(t) \rangle + \langle b(t) \rangle$

$$\langle g \rangle \equiv \langle g(t) \rangle \equiv \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{g}(t)dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{g}(t)dt \qquad (35)$$

(b) The **inner-product** operator:

Parseval

$$\langle g_1, g_2 \rangle \equiv \langle g_1(t), g_2(t) \rangle = \int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt \qquad (36)$$

$= \langle G_1, G_2 \rangle$

4.16. Using the above definition, we may write

- $E_g = \langle g, g \rangle = \langle G, G \rangle$ where $G = \mathcal{F}\{g\}$
- $P_g = \left\langle |g|^2 \right\rangle$

$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} g(t) g^*(t) dt$

$z z^* = |z|^2$

$= \langle g, g \rangle$

Inner Product (Cross Correlation)

Complex conjugate

- Vector

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y}^* = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}^* = \sum_{k=1}^{n} x_k y_k^*$$

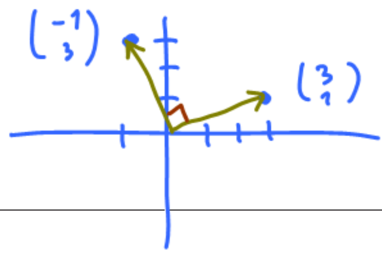
$$\left\langle \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\rangle = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix}^* = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = (-1)(3) + (3)(1) = 0$$

- Waveform: Time-Domain

$$\langle x, y \rangle = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$$

- Waveform: Frequency Domain

$$\langle X, Y \rangle = \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, df$$



Orthogonality

- Two signals are said to be **orthogonal** if their **inner product** is zero.
- The symbol $\perp$ is used to denote orthogonality.

Vector:

$$\langle \vec{a}, \vec{b} \rangle = \vec{a} \cdot \vec{b}^* = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}^* = \sum_{k=1}^{n} a_k b_k^* = 0$$

Time-domain:

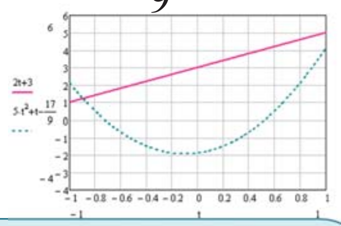
$$\langle a, b \rangle = \int_{-\infty}^{\infty} a(t)\, b^*(t)\, dt = 0$$

Frequency domain:

$$\langle A, B \rangle = \int_{-\infty}^{\infty} A(f)\, B^*(f)\, df = 0$$

Example:

$$2t + 3 \text{ and } 5t^2 + t - \frac{17}{9} \text{ on } [-1, 1]$$



Example (Fourier Series):

$$\sin\left(2\pi k_1 \frac{t}{T}\right) \text{ and } \cos\left(2\pi k_2 \frac{t}{T}\right) \text{ on } [0, T]$$

$$e^{j2\pi n \frac{t}{T}} \text{ on } [0, T]$$

Important Properties

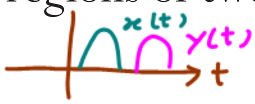
- Parseval's theorem

$$\langle x,y\rangle \equiv \int_{-\infty}^{\infty} x(t)\,y^*(t)\,dt = \int_{-\infty}^{\infty} X(f)\,Y^*(f)\,df \equiv \langle X,Y\rangle$$

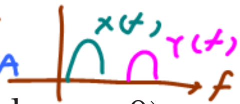
It is therefore sufficient to check only on the "convenient" domain.

$$x(t)\perp y(t) \quad \text{iff} \quad X(f)\perp Y(f).$$

- Useful observation: If the non-zero regions of two signals
  - TDM, TDMA ⇐ do not overlap in time domain or

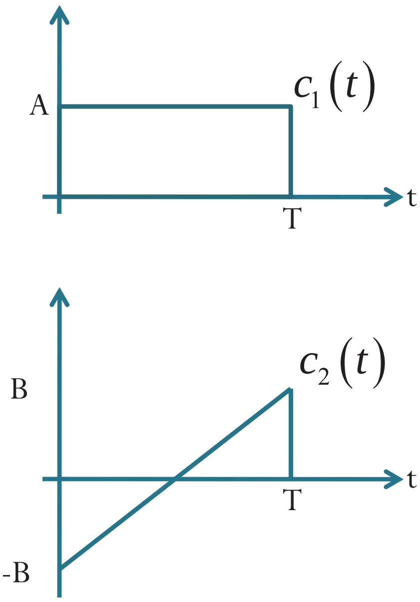


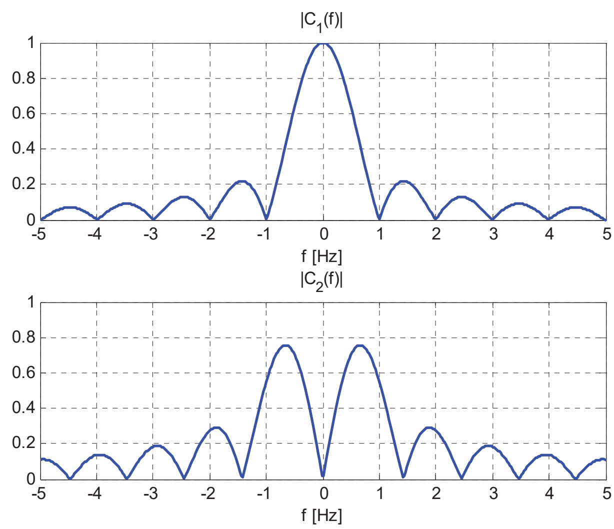
  - do not overlap in frequency domain, ⇒ FDM, FDMA



  then the two signals are orthogonal (their inner product = 0).
- However, in general, orthogonal signals may overlap both in time and in frequency domain.

Orthogonality: Example 1





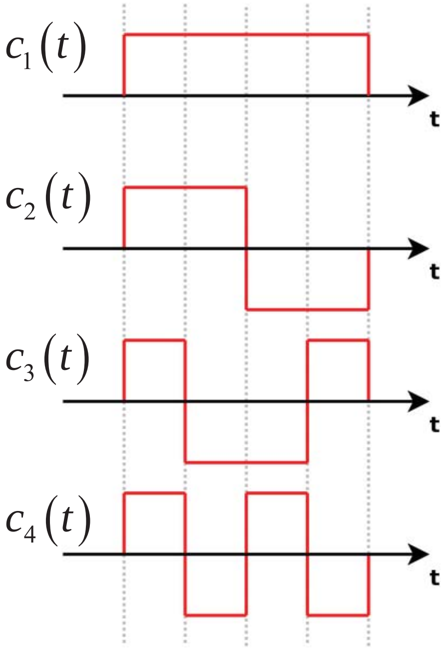
[CDMAEx.m]

The two waveforms above overlaps both in time domain and in frequency domian.

2G : CDMA
3G : WCDMA
code

Orthogonality: Example 2

An example of four "mutually orthogonal" signals.



When $i \neq j$,

$$\langle c_i(t), c_j(t) \rangle = 0$$

- Parseval's theorem: $\langle g_1, g_2\rangle = \langle G_1, G_2\rangle$
  where $G_1 = \mathcal{F}\{g_1\}$ and $G_2 = \mathcal{F}\{g_2\}$

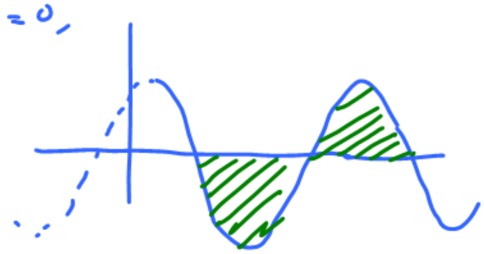
4.17. Time-Averaging over Periodic Signal: For ***periodic*** signal $g(t)$ with period T_0, the time-average operation in (35) can be simplified to

$$\langle g\rangle = \frac{1}{T_0}\int_{T_0} g(t)dt$$

$\int_0^{T_0}$ or $\int_{-T_0/2}^{+T_0/2}$ or $\int_{\alpha}^{\alpha+T_0}$

where the integration is performed over a period of g.

Example 4.18. $\langle\cos(2\pi f_0 t + \theta)\rangle = \begin{cases} 0, & f_0 \neq 0, \\ \cos\theta, & f_0 = 0, \end{cases}$



Similarly, $\langle\sin(2\pi f_0 t + \theta)\rangle = \begin{cases} 0, & f_0 \neq 0, \\ \sin\theta, & f_0 = 0. \end{cases}$

Example 4.19. $\left\langle\cos^2(2\pi f_0 t + \theta)\right\rangle =$

$$\begin{cases} \left\langle \frac{1}{2}\left(1 + \cos(2\pi(2f_0)t + 2\theta)\right)\right\rangle = \frac{1}{2}, & f_0 \neq 0 \\ \cos^2\theta, & f_0 = 0 \end{cases}$$

Example 4.20. $\left\langle e^{j(2\pi f_0 t + \theta)}\right\rangle = \langle\cos(2\pi f_0 t + \theta) + j\sin(2\pi f_0 t + \theta)\rangle$

$$= \begin{cases} 0, & f_0 \neq 0, \\ e^{j\theta}, & f_0 = 0 \end{cases}$$

Example 4.21. Suppose $g(t) = ce^{j2\pi f_0 t}$ for some (possibly complex-valued) constant c and (real-valued) frequency f_0. Find P_g.

$$P_g = \left\langle |g(t)|^2\right\rangle = \left\langle |c|^2 \left|e^{j2\pi f_0 t}\right|^2\right\rangle = \left\langle |c|^2\, 1\right\rangle = \left\langle |c|^2\right\rangle = |c|^2$$

4.22. When the signal $g(t)$ can be expressed in the form $g(t) = \sum_k c_k e^{j2\pi f_k t}$ and the f_k are distinct, then its (average) power can be calculated from

$$P_g = \sum_k |c_k|^2$$

the delta functions do not overlap in the freq domain.

"power of the sum = sum of power"

$P_\Sigma = \Sigma P$

when we have orthogonal signals.

$f_1 \neq f_2$

Example 4.23. Suppose $g(t) = \underset{c_1}{2}e^{j6\pi t} + \underset{c_2}{3}e^{j8\pi t}$. Find P_g.

$P_g = |c_1|^2 + |c_2|^2 = 2^2 + 3^2 = 4 + 9 = 13$

$f_1 = f_2 \Rightarrow P_\Sigma \neq \Sigma P$

Example 4.24. Suppose $g(t) = 2e^{j6\pi t} + 3e^{j6\pi t}$. Find P_g.

$g(t) = 5e^{j6\pi t}$

$P_g = 5^2 = 25$

Example 4.25. Suppose $g(t) = \cos(2\pi f_0 t + \theta)$. Find P_g.

Here, there are several ways to calculate P_g. We can simply use Example 4.19. Alternatively, we can first decompose the cosine into complex exponential functions using the Euler's formula:

$f_c = 0$: $g(t) = \cos(\theta) \Rightarrow P_g = \langle |g(t)|^2 \rangle = \cos^2\theta$

$\cos(2\pi f_c t + \theta)$

$f_c \neq 0$: $\frac{1}{2}e^{j(2\pi f_c t + \theta)} + \frac{1}{2}e^{-j(2\pi f_c t + \theta)} = \underbrace{\frac{1}{2}e^{j\theta}}_{c_1} e^{j2\pi \overset{f_1}{f_c} t} + \underbrace{\frac{1}{2}e^{-j\theta}}_{c_2} e^{-j2\pi \overset{f_2}{f_c} t}$

$P_g = \left|\frac{1}{2}e^{j\theta}\right|^2 + \left|\frac{1}{2}e^{-j\theta}\right|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$f_1 = f_c$, $f_2 = -f_c$, $\neq$

4.26. The (average) power of a sinusoidal signal $g(t) = A\cos(2\pi f_0 t + \theta)$ is

$$P_g = \begin{cases} \frac{1}{2}|A|^2, & f_0 \neq 0, \\ |A|^2\cos^2\theta, & f_0 = 0. \end{cases}$$

This property means any sinusoid with nonzero frequency can be written in the form

$$g(t) = \sqrt{2P_g}\cos(2\pi f_0 t + \theta).$$

4.27. Extension of 4.26: Consider sinusoids $A_k \cos(2\pi f_k t + \theta_k)$ whose frequencies are positive and distinct. The (average) power of their sum

$$g(t) = \sum_k A_k \cos(2\pi f_k t + \theta_k)$$

is

$$P_g = \frac{1}{2}\sum_k |A_k|^2.$$

For a signal $g(t)$,

Parseval

inner-product

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2\,dt = \langle g, g\rangle = \langle G, G\rangle$$

time average

$$P_g = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |g(t)|^2\,dt = \langle |g(t)|^2 \rangle$$

periodic signal with period T_0

$$= \frac{1}{T_0}\int_{T_0} |g(t)|^2\,dt$$

$$\int_{-T_0/2}^{T_0/2}, \quad \int_{0}^{T_0}, \quad \int_{\alpha}^{\alpha+T_0}$$

$f_1 \neq f_2$

Example 4.28. Suppose $g(t) = \overset{A_1}{2}\cos\left(2\pi\sqrt{3}t\right) + \overset{A_2}{4}\cos\left(2\pi\sqrt{5}t\right)$. Find P_g.

$f_1 \neq f_2$ $\qquad P_g = \frac{|A_1|^2}{2} + \frac{|A_2|^2}{2} = \frac{1}{2}\left(2^2+4^2\right) = 10$

Example 4.29. Suppose $g(t) = \underbrace{3\cos(2t)}_{f_1} + \underbrace{4\cos(2t-30°)}_{f_2} + \underbrace{5\sin(3t)}_{f_3}$. Find P_g.

$\Updownarrow \qquad \Updownarrow$

$3\angle 0° \; + \; 4\angle -30° \; \approx 6.77\angle -17.2°$

$4\times\frac{\sqrt{3}}{2} = 2\sqrt{3}$

$= \sqrt{2^2+(3+2\sqrt{3})^2}\;\angle\theta°$

$= A\angle\theta°$

$\Leftrightarrow A\cos(2t+\theta°)$

$P_g = \frac{A^2}{2} + \frac{5^2}{2} = 25 + 6\sqrt{3}$

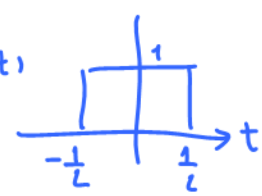
4.30. For ***periodic*** signal $g(t)$ with period T_0, there is also no need to carry out the limiting operation to find its (average) power P_g. We only need to find an average carried out over a single period:

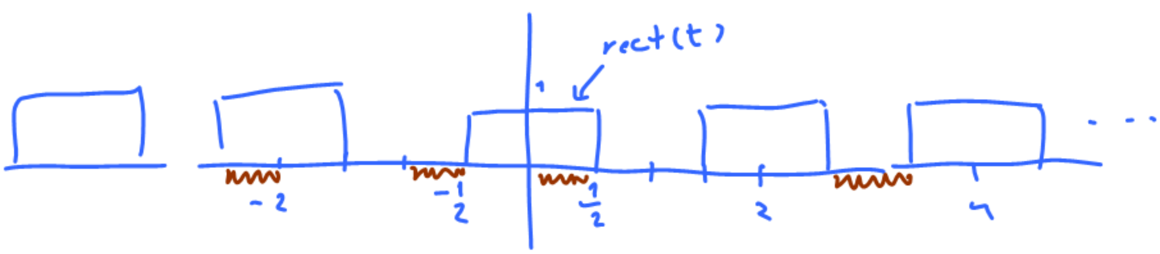
$$P_g = \frac{1}{T_0}\int_{T_0} |g(t)|^2 dt.$$

Example 4.31.

$g(t) = \sum_{n=-\infty}^{\infty} \text{rect}(t-2n)$

rect(t)





$E_g = \infty$

$\Rightarrow$ g(t) is not an energy signal

$P_g = \frac{1}{2}\int_{-1}^{1}|g(t)|^2 dt = \frac{1}{2}\int_{-1/2}^{1/2} 1^2 dt = \frac{1}{2}\times 1 = \frac{1}{2} \in (0,\infty) \Rightarrow$ g(t) is a power signal

4.32. When the Fourier series expansion (to be reviewed in Section 4.3) of the signal is available, it is easy to calculate its power:

(a) When the corresponding Fourier series expansion $g(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$ is known,

$$P_g = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

(b) When the signal $g(t)$ is real-valued and its (compact) trigonometric Fourier series expansion $g(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(2\pi k f_0 t + \angle\phi_k)$ is known,

$$P_g = c_0^2 + 2\sum_{k=1}^{\infty} |c_k|^2.$$

Definition 4.33. Based on Definitions 4.12 and 4.14, we can define three distinct classes of signals:

Ex. 

(a) If E_g is finite and nonzero, g is referred to as an ***energy signal***.

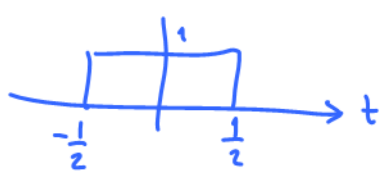
(b) If P_g is finite and nonzero, g is referred to as a ***power signal***.

(c) Some signals[17] are neither energy nor power signals.

Ex. 

- Note that the power signal has infinite energy and an energy signal has zero average power; thus the two categories are disjoint.

Example 4.34. Rectangular pulse

$g(t) =$ 

$\parallel$

rect(t)

$$E_g = \int_{-\infty}^{\infty} g^2(t)\, dt$$

$$= \int_{-1/2}^{1/2} 1^2\, dt = 1 \leftarrow \text{finite, nonzero}$$

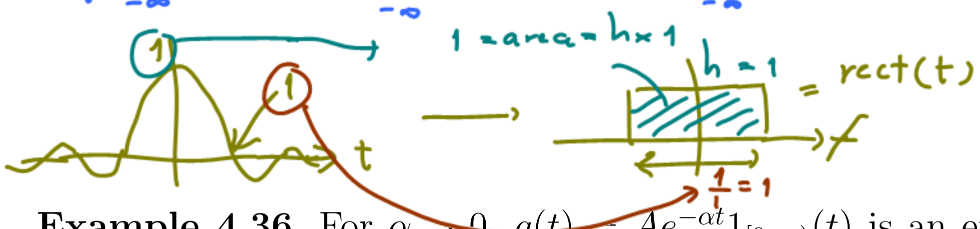
g(t) is an energy signal

$$P_g = \langle |g(t)|^2 \rangle = 0$$

[17]Consider $g(t) = t^{-1/4} 1_{[t_0,\infty)}(t)$, with $t_0 > 0$.

Example 4.35. Sinc pulse $g(t) = \text{sinc}(\pi t)$

$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} \text{sinc}^2(\pi t)\, dt = \int_{-\infty}^{\infty} |G(f)|^2 df = 1 \in (0, \infty) \Rightarrow$ $g(t)$ is an energy signal $\Rightarrow P_g = 0$ $\Rightarrow g(t)$ is not a power signal.



Example 4.36. For $\alpha > 0$, $g(t) = Ae^{-\alpha t}1_{[0,\infty)}(t)$ is an energy signal with $E_g = |A|^2/2\alpha$.

Example 4.37. The rotating phasor signal $g(t) = Ae^{j(2\pi f_0 t + \theta)}$ is a power signal with $P_g = |A|^2$.

Example 4.38. The sinusoidal signal $g(t) = A\cos(2\pi f_0 t + \theta)$ is a power signal with $P_g = |A|^2/2$.

4.39. Consider the transmitted signal

$$x(t) = m(t)\cos(2\pi f_c t + \theta)$$

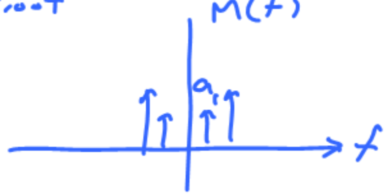
in DSB-SC modulation. Suppose $M(f - f_c)$ and $M(f + f_c)$ do not overlap (in the frequency domain).

(a) If $m(t)$ is a power signal with power P_m, then the average transmitted power is

$$P_x = \frac{1}{2}P_m.$$

"Proof"

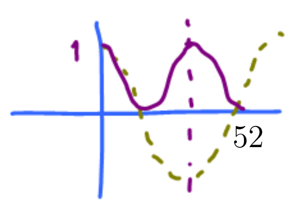
$a_1 e^{j2\pi f_1 t}$



- Q: Why is the power (or energy) reduced?

$x(t) = m(t)\cos(\quad)$

$x^2(t) = m^2(t)\cos^2(\quad)$



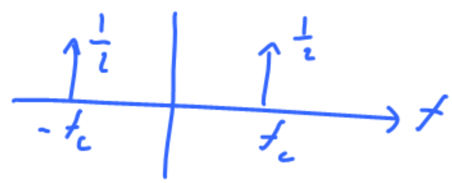
- Remark: When $x(t) = \sqrt{2}m(t)\cos(2\pi f_c t + \theta)$ (with no overlapping between $M(f - f_c)$ and $M(f + f_c)$), we have $P_x = P_m$.

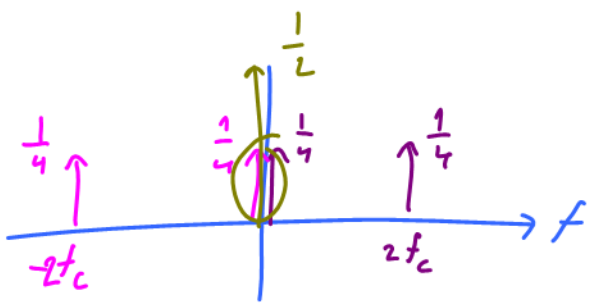
(b) If $m(t)$ is an energy signal with energy E_m, then the transmitted energy is

$$E_x = \frac{1}{2}E_m.$$

Example 4.40. Suppose $m(t) = \cos(2\pi f_c t)$. Find the average power in $x(t) = m(t)\cos(2\pi f_c t)$.

$P_m = \frac{|A|^2}{2} = \frac{1^2}{2} = \frac{1}{2}$





$P_x = \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2$

$= \frac{1}{16} \times 2 + \frac{1}{4} = \frac{1}{8} + \frac{1}{4} = \frac{3}{8} \left(\neq \frac{1}{2} P_m\right)$